Kinematics

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REFERENCE FRAME

Reference frame is a set of coordinate axis with respect to which we measure the position of any particle.

MOTION

A particle is said to have moved in a certain reference frame if its position vector changes.

DISPLACEMENT ($\vec{S}$)

The change in position vector is called the displacement.
Mathematically,

$$\vec{S} = \vec{r}_f - \vec{r}_{\hat{i}} \qquad (r_f = \text{final position})$$

$$\vec{S} = \Delta\vec{r} \qquad (r_i = \text{initial position})$$

$(\Delta = \text{change})$

Q.) A final coordinate = (3,4,5) and it has undergone a displacement of ($\vec{S} = 2\hat{i} - 15\hat{j} + \hat{k}$). Find its initial coordinates.

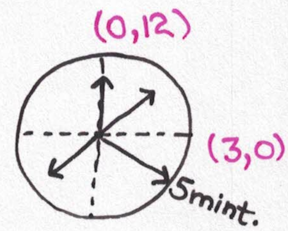
$$\hat{r_i} = \vec{r}_f - \vec{S}$$

$$\hat{r_i} = (3\hat{i} + 4\hat{j} + 5\hat{j}) - (2\hat{i} - 15\hat{j} + \hat{k})$$

$$\hat{r_i} = \hat{i} + 19\hat{j} + 4\hat{k}$$

Q.) The minute hand of a clock tower is ½ m. long and time is 12 o'clock. Take y-axis along 12 o'clock position and find $\vec{S}$ of its tip in

(i) 10 min.
(ii) 20 min.
(iii) 40 min.



(i) $\hat{r_i} = \frac{1}{2}\hat{j}$

$$\vec{r}_f = \frac{1}{2}\sin 60^\circ\,\hat{i} + \frac{1}{2}\cos 60^\circ\,\hat{j}$$

$$\vec{r}_f = \frac{1}{2} \times \frac{\sqrt{3}}{2}\hat{i} + \frac{1}{4}\hat{j}$$

$$\vec{S} = \frac{\sqrt{3}}{4}\hat{i} - \frac{1}{4}\hat{j}$$

(ii) $\vec{r}_f = -\frac{1}{2}\sin 30^\circ \hat{j} + \frac{1}{2}\cos 30^\circ \hat{i}$

$= -\frac{1}{4}\hat{j} + \frac{\sqrt{3}}{4}\hat{i}$

$\vec{S} = -\frac{3}{4}\hat{j} + \frac{\sqrt{3}}{4}\hat{i}$

(iii) $\vec{r}_f = \frac{1}{2}(-\sin 60^\circ)\hat{i} - \frac{1}{2}\cos 60^\circ \hat{k}$

$= -\frac{1}{2}\times\frac{\sqrt{3}}{2}\hat{i} - \frac{1}{4}\hat{k}$

$\vec{S} = -\frac{\sqrt{3}}{4}\hat{i} - \frac{3}{4}\hat{k}$

DISTANCE (ℓ)

The length of the path in moving from one point to another is called the distance.

Q.) Find distance travelled in previous question.

$\frac{\pi}{3}\times\frac{1}{2} = \frac{\pi}{6}$ m.

NOTE: Distance is always greater than or equal to magnitude of displacement.

AVERAGE VELOCITY ($\vec{V}_{avg}$)

The ratio of displacement during a time interval to the duration of that time interval is known as average velocity for that time interval.

Mathematically, $V_{avg.} = \frac{\Delta\vec{S}}{\Delta t} = \frac{\Delta\vec{r}}{\Delta t}$

Q.) Second's hand of a clock is 0.5m long. Find avg. velocity of the tip in
(i) 10 sec.
(ii) 15 sec.
(iii) 40 sec.

(i) $\vec{r}_i = \frac{1}{2}\hat{j}$

$\vec{r}_f = \frac{1}{2}\sin 60^\circ\hat{i} + \frac{1}{2}\cos 60^\circ\hat{j}$

$= \frac{\sqrt{3}}{4}\hat{i} + \frac{1}{4}\hat{j}$

$\vec{v} = \frac{\vec{S}}{t} = \frac{\frac{\sqrt{3}}{4}\hat{i} - \frac{1}{4}\hat{j}}{10} = \frac{\sqrt{3}}{40}\hat{i} - \frac{1}{40}\hat{j}$

(ii) $\vec{r}_f = \frac{1}{2}\hat{i}$

$\vec{S} = \frac{1}{2}\hat{i} - \frac{1}{2}\hat{j}$

$\vec{v} = \frac{1}{30}\hat{i} - \frac{1}{30}\hat{j}$

(iii) From previous qus.,

$\vec{S} = -\frac{\sqrt{3}}{4}\hat{i} - \frac{3}{4}\hat{j}$

$\vec{v} = -\frac{\sqrt{3}}{160}\hat{i} - \frac{3}{160}\hat{j}$

Q. A particle travels along a straight line such that its position as a function of time is given by $x = f(t) = t^2$ where x is the coordinate of point and t is time (s). Find average velocity during time interval.

(i) $t_1 = 2$ & $t_2 = 3$

(ii) $t_1 = 2$ & $t_2 = 2.1$

(iii) $t_1 = 2$ & $t_2 = 2.01$

(iv) $t_1 = 2$ & $t_2 = 2.001$

(v) What do you think will be the answer if we keep on making the interval arbitrary small.

(i) $v_{avg.} = \frac{5}{1} = 5 \, m/s$

(ii) $v = \frac{0.41}{0.1} = 4.1 \, m/s$

(iii) $v = \frac{0.0401}{0.01} = 4.01 \, m/s$

(iv) $v = \frac{0.004001}{0.001} = 4.001 \, m/s$

(v) instantaneous velocity $= 4 \, m/s$

DIFFERENTIATION

Let x be some function of time then at any general instant of time we can talk about the ratio of change in x to the change in time for any arbitrarily small interval of time. This ratio is symbolically represented by the symbol.

$$\left(\frac{dx}{dt}\right)$$

and is read by several manner :-

- differentiation of x with respect to t.
- derivative of x with respect to t.
- instantaneous rate of change of x with respect to t.
- slope of the x-t graph.

Mathematically, the differentiation can be defined as,

$$\frac{dx}{dt} = \lim_{\Delta t \to \Delta} \left(\frac{\Delta x}{\Delta t}\right)$$

The position x of a particle varies with time as $x = t^2$. Find the instantaneous rate of change of x with respect to general interval t.

$$\frac{dx}{dt} = \lim_{\Delta t \to 0} \frac{(t+\Delta t)^2 - t^2}{\Delta t}$$

$$= \lim_{\Delta t \to 0} \frac{\cancel{t^2} + 2\Delta t . t + \Delta t^2 - \cancel{t^2}}{\Delta t}$$

$$= \lim_{\Delta t \to 0} 2t + \Delta t$$

$$= 2t \qquad (\Delta t \approx 0)$$

Q.) A particle moves according to $x = t^3$. Using the basic definition of differentiation find the instantaneous velocity of a particle at a general time t.

$$\frac{dx}{dt} = \lim_{\Delta t \to 0} \frac{\Delta x}{\Delta t}$$

$$= \lim_{\Delta t \to 0} \frac{(t+\Delta t)^3 - t^3}{\Delta t}$$

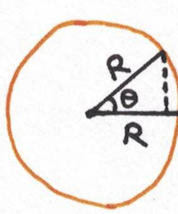
$$= \lim_{\Delta t \to 0} \frac{\cancel{t^3} + \Delta t^3 + 3t^2\Delta t + 3t\Delta t^2 - \cancel{t^3}}{\Delta t}$$

$$= \lim_{\Delta t \to 0} \Delta t^2 + 3t^2 + 3t\Delta t$$

$$= 3t^2 \qquad (\Delta t \approx 0)$$

$$\lim_{\theta \to 0} \sin\theta = \theta$$

$$\sin\theta = \frac{\cancel{R}\theta}{\cancel{R}} = \theta$$



Chord is equal to arc

1.) $\sin\theta \approx \tan\theta \approx \theta$

2.) $\cos\theta \approx 1$ if $\lim_{\Delta\theta \to 0}$

$$\frac{dx}{dt} = \lim_{\Delta t \to 0} \frac{\sin(t+\Delta t) - \sin t}{\Delta t}$$

$$= \frac{\sin t \cdot \cos\Delta t + \cos t \cdot \sin\Delta t - \sin t}{\Delta t}$$

$$= \frac{\sin t + \cos\Delta t - \sin t}{\Delta t}$$

$$= \cos t$$

$$\left\{ \begin{array}{l} \sin\Delta t = \Delta t \\ \cos\Delta t = 1 \end{array} \right.$$

LIST OF STANDARD DIFFERENTIATION FORMULAE

	x	$\frac{dx}{dt}$
(1)	t^n	$n\,t^{n-1}$
(2)	$\sin(t)$	$\cos(t)$
(3)	$\cos(t)$	$-\sin(t)$
(4)	$\tan(t)$	$\sec^2(t)$
(5)	$\sec(t)$	$\sec(t)\tan(t)$
(6)	$\mathrm{cosec}(t)$	$-\mathrm{cosec}(t)\cot(t)$
(7)	$\cot(t)$	$-\mathrm{cosec}^2(t)$
(8)	e^t	e^t
(9)	$\ln(t)$	$1/t$
(10)	C (Constant)	0

RULES OF DIFFERENTIATION

(1) Derivative of a sum = Sum of derivatives

Eg – $x = t^2 + t^3$

$$\frac{dx}{dt} = 2t + 3t^2$$

(2) A multiplied constant can be taken outside the derivative.

Eg – $$\frac{d}{dt}(3t^2) = 3\frac{d}{dt}(t^2) = 3.2t = 6t$$

(3) Product Rule –

$$\frac{d}{dt}(u.v) = u\frac{dv}{dt} + v\frac{du}{dt}$$

Eg– (i) $x = t^2 \sin(t)$

$$\frac{dx}{dt} = t^2 \cos t + \sin t . 2t$$

(ii) $x = e^t \ln(t)$

$$\frac{dx}{dt} = e^t \times \frac{1}{t} + \ln(t)e^t$$
$$= e^t\left(\frac{1}{t} + \ln(t)\right)$$

(4) **Quotient Rule:**

$$\frac{d}{dt}\left(\frac{u}{v}\right) = \frac{v\frac{du}{dt} - u\frac{dv}{dt}}{v^2}$$

Eg - $\frac{d}{dt}\left(\frac{1}{\operatorname{Cosec} t}\right) = -\frac{(-\operatorname{Cosec} t \cdot \operatorname{Cot} t)}{\operatorname{Cosec}^2 t}$

$$= \frac{\operatorname{Cot} t}{\operatorname{Cosec} t}$$
$$= \operatorname{Sin} t \cdot \operatorname{Cot} t$$
$$= \operatorname{Cos} t$$

(5) **Chain rule:**

If $y = f[g(x)]$, then

$$\frac{dy}{dx} = \frac{df}{dg} \times \frac{dg}{dx}$$

Eg - (i) $\frac{d}{dt} e^{\sin(\sqrt{3t^2+2t+4})}$

$$= e^{\sin\sqrt{3t^2+2t+4}} \times \operatorname{Cos}\sqrt{3t^2+2t+4} \times \frac{1}{2}(3t^2+2t+4)^{-1/2} \times (6t+2)$$

(ii) $\frac{d}{dt}\left(\frac{1}{\sqrt{\sin^2 t + 2t}}\right)$

$$= -\frac{1}{2}(\sin^2(t) + 2t)^{-3/2} \times (2\operatorname{Sin}t \operatorname{Cos}t + 2)$$

(iii) $\frac{d}{dt}\left(\frac{1}{\sin\sqrt{t^2+1}}\right)$

$$= -(\sin\sqrt{t^2+1})^{-1/2} \times \operatorname{Cos}(\sqrt{t^2+1}) \times \frac{1}{2\sqrt{t^2+1}} \times 2t$$

(If $y = t^2$,
what will be $\frac{dy}{dx}$?)

$$\Rightarrow \frac{dy}{dx} = 2t \times \frac{dt}{dx}$$

Q.) $y = (t-a)(t-b)$ where a & b are constants. Find $\frac{dy}{dx}$.

$= (t-a)(\;) + (t-b)(1)$

$= 2t - a - b$

Q.) $y = (t-4)(3t^2-4t-5)$. Find dy/dx

$= (t-4)(6t-4) + (3t^2-4t-5)(1)$

$= 6t^2 - 4t - 24t + 16 + 3t^2 - 4t - 5$

$= 9t^2 - 32t + 11$

Q.) $y = 2\sqrt{\cot t^2}$. Find dy/dx

$= \frac{1}{\sqrt{\cot t^2}} \times -\operatorname{cosec} t^2 \times 2t$

Q.) Find the rate of change of area of a circle if its instantaneous radius is 5cm increasing at a rate of 1cm per second.

$A = \pi r^2$

$\frac{dA}{dt} = \pi \times 2r \times \frac{dr}{dt}$

$= 10\pi \ cm^2/s$

Q.) Repeat the previous que. for volume of a sphere.

$V = \frac{4}{3}\pi r^3$

$\frac{dV}{dt} = \frac{4}{3}\pi \times 3r^2 \times \frac{dr}{dt}$

$= \frac{300}{3}\pi = 100\pi \ cm^2/s$

Q.) A stone is dropped in a quite lake. A ripple wave moves at 4cm/s. How fast is the enclosed area in the ripple increases when the radius is 10cm.

$A = \pi r^2$

$\frac{dA}{dt} = 2\pi r \times 4$

$= 20\pi \times 4$

$= 80\pi$

Q.) The length x of a rectangle is decreasing at a rate of 3cm/min. & its width y is increasing at a rate of 2cm/min. when x is 10 cm & y is 6cm find the rate of change of perimeter & area.

$P = 2(l+b)$

$$\frac{dP}{dt} = \frac{d}{dt}2l + \frac{d}{dt}2b$$

$$= 2\times -3 + 2\times 2$$

$$= -2$$

$$\frac{dA}{dt} = \frac{d}{dt}(lb)$$

$$= l\frac{db}{dt} + b\frac{dl}{dt}$$

$$= 20 - 18$$

$$= 2$$

Q.) Find the rate of change of volume of a Cone if radius & height is changing with time.

$$V = \frac{1}{3}\pi r^2 h$$

$$\frac{dV}{dt} = \frac{1}{3}\pi\left(\frac{d(r^2h)}{dt}\right)$$

$$= \frac{1}{3}\pi \times \left(r^2\frac{dh}{dt} + h\frac{dr^2}{dt}\right)$$

Q.) Find the rate of change of surface area & the volume of a cube of edge 'a' if a is changing with time.

$$V = a^3$$

$$\frac{dV}{dt} = \frac{da^3}{dt} \times 3a^2\frac{da}{dt}$$

$$A = 6a^2$$

$$\frac{dA}{dt} = 12a \times \frac{da}{dt}$$

Q.) The volume of a cube is increasing at a rate of 9 cm³/s. How fast is its surface area increasing when the length of edge is 10 cm.

$$\frac{dV}{dt} = 9$$

$$\frac{da^3}{dt} = 9$$

$$3a^2 \times \frac{da}{dt} = 9$$

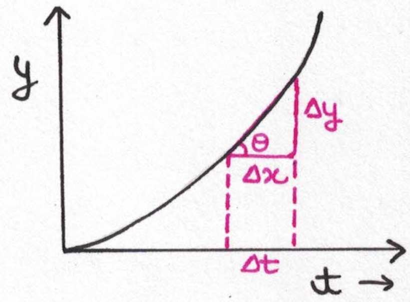
$$\frac{da}{dt} = \frac{9}{300}$$

$$\frac{6da^2}{dt} = 6\times 2a \times \frac{da}{dt}$$

$$= 20\cancel{0} \times \frac{9}{30\cancel{0}}$$

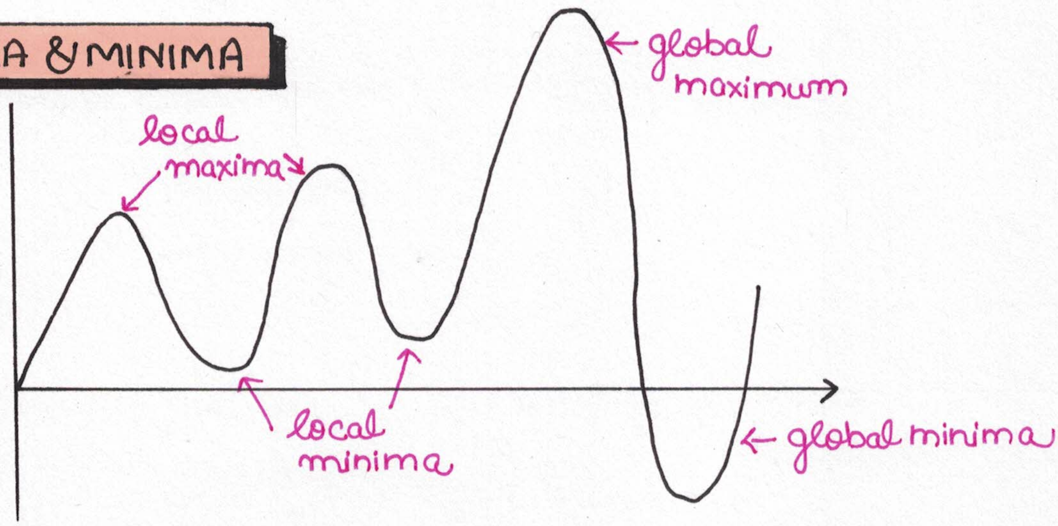
$$= \frac{18}{\cancel{30}_{5}} \times \cancel{6} \quad = 3.6\ cm^2/sec$$

Graphically, the derivative is the tan of the angle which the tangent of the graph makes with the positive horizontal axis. This is also called slope of graph.



$$\frac{dy}{dt} = \tan\theta$$

MAXIMA & MINIMA



For any smooth curve without slope kinks, we see that the tangent of the curve becomes parallel to the horizontal axis at every peak and valley. Thus to locate the maxima & minima of any function its derivative should be equated to zero.

i.e. $\frac{dy}{dt} = 0$

Together maxima and minima are called the extrema.

PRACTICE TIME

1:) Find value of x for which y is extreme.

$y = x^2 + 2x + 4$

$\frac{dy}{dx} = 0 = 2x + 2$

$\Rightarrow x = -1$

$y = 1 - 2 + 4$
$= 3$

or $y = (x+1)^2 + 3$

It will be minimum when $x = -1$, $\Delta y = 3$

2.) $y = \sin x + 4$. Find extreme

$$\frac{dy}{dx} = 0 = \cos x$$

$$x = (2\pi+1)\frac{\pi}{2}$$

SECOND DERIVATIVE

Derivative of a derivative is called the second derivative.

Eg:- $y = t^3$

$$\frac{dy}{dt} = 3t^2$$

$$\frac{d^2y}{dt^2} = 6t \quad \text{(second derivative)}$$

DIFFERENTIATING BETWEEN MAXIMA & MINIMA

As we approach a peak, moving in the direction of positive x-axis the slope of the graph keeps on decreasing i.e. the rate of change of slope is negative or the second derivative of the function is negative.

When we approach a minimum, the slope keeps on increasing and thus the rate of change of slope is positive i.e. $d^2y/dt^2 > 0$.

As dy/dt is called the slope, d^2y/dt^2 is called Curvature (Katora).

Q.) Find extrema of $y = -3t^2 + 2t + 6$

$$\frac{d^2y}{dt^2} = \frac{dy}{dt}(-6t+2)$$

$$\Rightarrow -6t+2 = 0$$

$$\Rightarrow t = 1/3$$

$$\frac{d^2y}{dt^2} = -6$$

$$y_{max} = -\cancel{3} \times \frac{1}{\cancel{9}_3} + 2 \times \frac{1}{3} + 6$$

$$y_{max} = \frac{1}{3} + 6$$

PRACTICE TIME

Q.1.) A field along a river is to be fenced around 3 sides using a length of rope 20m. What is the max. area & the corresponding dimension of the field?

$$\text{Area} = (20-2x)(x) = -2x^2 + 20x$$

$$\frac{dA}{dx} = 0 = -2x^2 + 20x$$
$$0 = -4x + 20$$
$$\boxed{x = 5}$$

$$\frac{d^2A}{dx^2} = -4 \quad (-ve), \text{ so it is maxima}$$
$$A = 5 \times 10 = 50 \text{ m}^2$$

Q.2.) Find maxima & minima of $y = \frac{x^3}{3} - \frac{3x^2}{2} + 2x$

$$\frac{dy}{dx} = 3x^2 - 3x + 2 = 0$$
$$(x-2)(x-1) = 0$$
$$x = 2, \; x = 1$$

$$\frac{d^2y}{dx^2} = 2x - 3$$

for ① $\quad 4 - 3 = 1 \quad (+ve) \Rightarrow$ minima

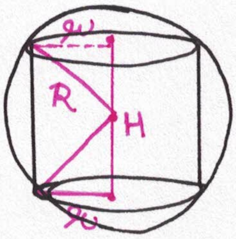
for ② $\quad 2 - 3 = -1 \quad (-ve) \Rightarrow$ maxima

$$y_{max} = 3 - 3 + 2 = 2$$
$$y_{min} = 4 - 12 + 2 = -6$$

Q.3) What is the max. volume of a cylinder which can be fitted in a sphere of radius R. Let height be 2H.

$$V = \pi (R^2 - H^2) \times 2H$$
$$= 2\pi R^2 H - 2\pi H^3$$

$$\frac{dV}{dH} = 2\pi R^2 - 6\pi H^2 = 0$$
$$\Rightarrow 2\pi (R^2 - 3H^2) = 0$$
$$\Rightarrow R^2 = 3H^2$$
$$R = \sqrt{3} H$$
$$H = R/\sqrt{3}$$



$$V = 2\pi \left(R^2 \times \frac{R}{\sqrt{3}} - \frac{R^3}{3\sqrt{3}} \right)$$
$$= \frac{4\pi R^3}{3\sqrt{3}}$$

Q.4.) What is the maximum volume of a cone that can be described in a sphere of R.

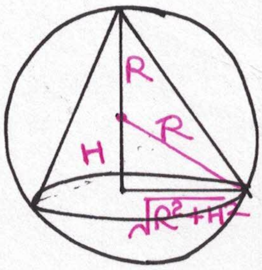
$$V = \frac{1}{3}\pi \times R^2 - H^2 \times (R + H) = \frac{1}{3}\pi (R^3 + HR^2 - H^2R - H^3)$$

$$\frac{dv}{dH} = \frac{1}{3}\pi(R^2 - 2HR - 3H^2) = 0$$

$$3H^2 + 2HR - R^2 = 0$$

$$(H+1)(3H-R) = 0$$

$$H = -1 \quad \text{or} \quad H = \frac{R}{3}$$



$$v = \frac{1}{3}\pi \times \frac{8}{9}R^2 \times \frac{4}{3}R$$

$$v = \frac{32}{81}\pi R^3$$

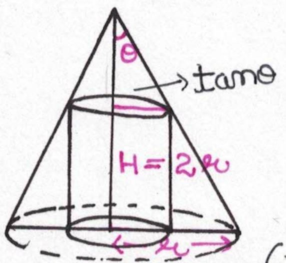
Q.5) what is the maximum Curved surface area of a cylinder which can be put inside a cone of radius 'r' and height '2r'.

$$\text{Curved Surface area} = 2\pi(2R-H) \times \frac{\overset{2}{\cancel{4}}}{\cancel{2}}H$$

$$S = 4\pi(2RH - H^2)$$

$$\frac{dS}{dt} = 4\pi(2R - 2H) = 0$$

$$\therefore \boxed{H = R}$$



$$\left(\tan\theta = \frac{\text{radius}}{2r - H}\right)$$

$$\tan\theta = \frac{1}{2}$$

INTEGRATION (Anti - Derivatives)

Integration is the reverse process of differentiation. Physically, it means summation of infinitesimal parts. Graphically, it means area under the graph.

STANDARD RESULTS OF INTEGRATION

$$\int \cos(x).dx = \sin(x) + C$$

$$\int \sin(x).dx = -\cos(x) + C$$

$$\int e^x.dx = e^x + C$$

$$\int 1/x.dx = \ln(x) + C$$

$$\int \sec^2(x).dx = \tan(x) + C$$

$$\int \operatorname{cosec}^2(x)dx = -\cot(x) + C$$

$$\int x^n . dx = \frac{x^{n+1}}{n+1} + C$$

$$\int \frac{1}{a^2+x^2} . dx = \frac{1}{a}\left(\tan^{-1}\left(\frac{x}{a}\right)\right) + C$$

RULES FOR INTEGRATION

1.) A multiplied constant can be taken outside the integral.

Eg- $\int 3\cos(x).dx = 3\int \cos(x).dx$

$= 3\sin(x) + C$

2.) Integral of a sum is equal to the sum of the integrals.

Eg - $\int 3\cos(x) + 2\sin(x).dx = 3\sin x - 2\cos x + C$

3.) $\int f(ax+b)\,dx = \frac{1}{a}\int f(X).dX$

where $X = (ax+b)$

Eg - (i) $\int \cos(3x+4).dx = \frac{1}{3}\sin(3x+4) + C$

(ii) $\int (2x+3)^2.dx = \frac{(2x+3)^3}{3\times 2} + C = \frac{(2x+3)^3}{6} + C$

(iii) $\int e^{(3x+4)}.dx = \frac{e^{(3x+4)}}{3} + C$

(iv) $\int \frac{1}{2x+5}.dx = \frac{\ln(2x+5)}{2} + C$

(v) $\int \frac{1}{\sqrt{3x+4}}.dx = \frac{2\sqrt{3x+4}}{3} + C$

(vi) $\int \sin(4x+3)dx = -\frac{\cos(4x+3)}{4} + C$

DEFINITE INTEGRAL

If $\int f(x).dx = F(x)$

then $\int_a^b f(x).dx = F(b) - F(a)$

Eg: $\int_0^{\frac{\pi}{2}} \cos(x).dx = \sin 90^\circ - \sin 0^\circ$

$= 1$

$\int_0^{\frac{\pi}{4}} \cos(2x).dx = \frac{1}{2}\left[\sin 2x\right]_0^{\pi/4}$

$= \frac{1}{2}$

Eg: 1.) $\int_0^{\pi} 2t(\sin t^2).dt$

let $x = t^2$

$\frac{dx}{dt} = 2t$

$= \int \frac{dx}{\cancel{dt}} \sin x.\cancel{dt}$

at $t=0$, $x=0$
$t=\pi$, $x=\pi^2$

$= \int_0^{\pi^2} \sin x.dx$

$= [-\cos x]_0^{\pi^2}$

$= -\cos \pi^2 + \cos 0$

$= +1 - \cos \pi^2$

2.) $\int_1^2 t^3(1+t^4)^3.dt$

let $x = 1+t^4$

$\frac{dx}{dt} = 4t^3$

$dt.t^3 = \frac{dx}{4}$

$= \int_1^2 \frac{x^3}{4} dx$

$= \frac{1}{4}\int_1^2 x^3 dx$

$= \frac{1}{4}\left[\frac{x^4}{4}\right]_1^2 = 1 - \frac{1}{16}$

$= \frac{15}{16}$

3.) $\int_0^{\pi/2} \frac{\sin \omega}{(3+2\cos\omega)^2}.d\omega$

let $3+2\cos\omega = x$

$\frac{dx}{d\omega} = -2\sin\omega$

at $0 \Rightarrow 5$

$= \int_5^3 -\frac{dx}{2x^2}$

$= -\frac{1}{2}\int_5^3 \frac{1}{x^2}.dx$

$= -\frac{1}{2}\left[-\frac{1}{x}\right]_5^3$

$$= -\frac{1}{2}\left[-\frac{1}{3}+\frac{1}{5}\right]$$

$$= -\frac{1}{\cancel{2}}\left[-\frac{\cancel{2}}{15}\right] = \frac{1}{15}$$

4.) $\int_0^1 \frac{5r}{(4+r^2)^2} \cdot dr$

let $x = 4+r^2$

$$\frac{dx}{dr} = 2r \cdot dr$$

$$= \int_0^1 \frac{5r}{x^2} \cdot dr$$

$$= \int_4^5 \frac{5}{2x^2} \cdot dx$$

$$= \frac{5}{2}\left[\frac{1}{x}\right]_4^5$$

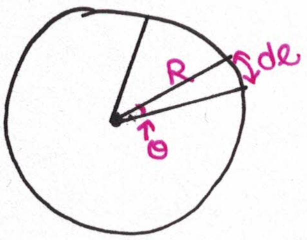
$$= -\frac{5}{2}\left[\frac{1}{5}-\frac{1}{4}\right] = \frac{1}{8}$$

Q.) Using definite integral, prove that circumference of a circle is $2\pi r$.

$$\int_0^L dl = \int_0^{2\pi} R \cdot d\theta$$

$$[l]_0^L = R\,[\theta]_0^{2\pi}$$

$$\boxed{L = 2\pi R}$$



Q.) Using definite integral, prove that area of a circle is πr^2.

$$dA = \frac{1}{2} R\,d\theta \times R$$

$$\int_0^A dA = \int_0^{2\pi} \frac{1}{2} R^2 \cdot d\theta$$

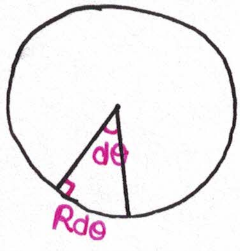
$$(A)_0^A = \frac{1}{2} R^2 (\theta)_0^{2\pi}$$

$$A = \frac{1}{2} R^2 (\theta)_0^{2\pi}$$

$$A = \frac{1}{\cancel{2}} R^2 \times \cancel{2}\pi$$

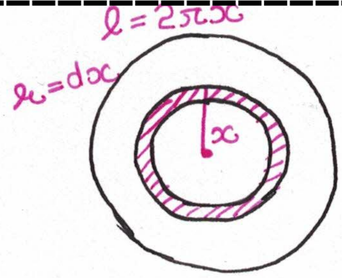
$$= \pi R^2$$

$$\boxed{A = \pi R^2}$$



OR

in the form of rings



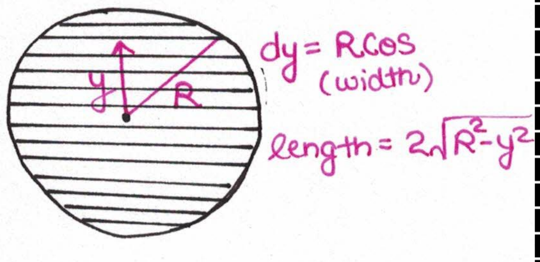
$$\int_0^A dA = \int_0^R 2\pi x . dx$$

$$[A]_0^A = 2\pi \left[\frac{x^2}{2}\right]_0^R$$

$$A = 2\pi \times \frac{R^2}{2} = \pi R^2$$

OR

in the form of ~~stri~~ strips



$$\int_0^A dA = 2\int_0^R \sqrt{R^2-y^2}. dy$$

let $y = R\sin\theta$

$dy = R\cos\theta . d\theta$

$$A = 2\int_0^{\pi/2} (R\cos\theta) \times R\cos\theta . d\theta$$

$$A = 2\int_0^{\pi/2} R^2 \cos^2\theta . d\theta$$

$$= 2R^2 \int_0^{\pi/2} \cos^2\theta . d\theta$$

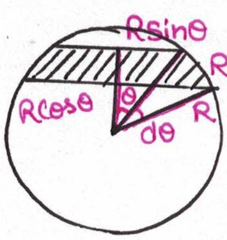
$$= \cancel{2}R^2 \int_0^{\pi/2} \frac{1+\cos 2\theta}{\cancel{2}} . d\theta$$

$$= R^2 \left[\theta + \frac{\sin 2\theta}{2}\right]_0^{\pi/2}$$

$$= R^2 \left[\pi/2\right]$$

$$= \frac{\pi R^2}{2} \quad \text{(area of semicircle)}$$

Q. Using integration, prove that surface area of a sphere is $4\pi r^2$.



$$\underset{\text{(Surface area)}}{dS} = \underset{\text{(length)}}{2\pi R\sin\theta} \times \underset{\text{(breadth)}}{R d\theta}$$

$$\int_0^S dS = \int_0^{\pi} 2\pi R \sin\theta \times R d\theta$$

$$S = 2\pi R^2 \int_0^{\pi} \sin\theta . d\theta$$

$$S = 2\pi R^2 [-\cos\theta]_0^{\pi}$$

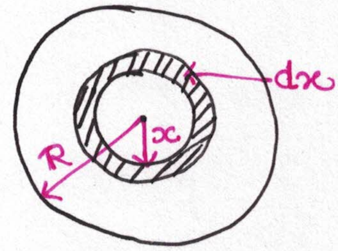
$$S = 2\pi R^2 [-\cos\pi + \cos 0]$$

$$\boxed{S = 4\pi R^2}$$

Q.) Using Integration, prove that volume of Sphere is $\frac{4}{3}\pi R^3$.

$$\int_0^v dv = \int_0^R 4\pi x^2 . dx$$

$$v = \frac{4}{3}\pi R^3$$



Q.) Using integration prove that CSA of Cone is $\pi R l$.

$$ds = 2\pi y \tan\theta . \sec\theta . dy$$

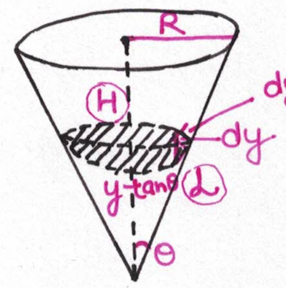
$$\int_0^S ds = \int_0^H 2\pi y \tan\theta \sec\theta dy$$

$$S = \int_0^H 2\pi \times y \times \frac{R}{H} \times dy \times \frac{l}{H}$$

$$= 2\pi \times \frac{R}{H} \times \frac{l}{H} \int_0^H y . dy$$

$$= \frac{\cancel{2}\pi R l}{\cancel{H^2}} \times \frac{\cancel{H^2}}{\cancel{2}}$$

$$= \pi R l$$



$H = \sqrt{l^2 - R^2}$

$$\left[\begin{array}{l} \tan\theta = \frac{R}{H} \\ \sec\theta = \frac{l}{H} \end{array} \right]$$

Q.) Using integration, prove that volume of Cone is $\frac{1}{3}\pi R^2 H$

$$dv = \pi y^2 \tan^2\theta . dy$$

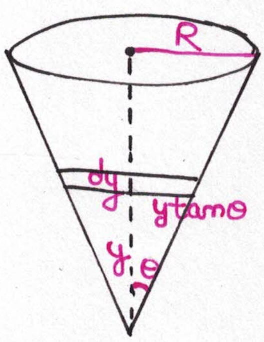
$$\int_0^v dv = \int_0^H dy . \pi y^2 \tan^2\theta$$

$$v = \pi \times \frac{R}{H} \times \frac{R}{H} \int_0^H y^2 . dy$$

$$= \pi \frac{R^2}{H^2} \times \left[\frac{y^3}{3} \right]_0^H$$

$$= \frac{\pi R^2}{\cancel{H^2}} \times \frac{\cancel{H^3}}{3}$$

$$= \frac{1}{3}\pi R^2 H$$



DIFFERENTIAL EQUATIONS

1st ORDER EQUATION

An equation of the form $dy/dx = f(y, x)$ is called a 1st order differential equation. The relation b/w y and x which satisfies the differential equation is called a solution of differential eq.^{n}

Q. Solve the differential equation $dy/dx = xy$.

$$\int \frac{dy}{y} = \int dx \cdot x$$

$$\log y = \frac{x^2}{2} + c$$

$$e^{(x^2/2 + c)} = y$$

$$y = e^c \times e^{x^2/2}$$

$$y = c \times e^{x^2/2}$$

Q. Solve $dy/dx = \left(\frac{2\cos x + 1}{y}\right)$

$$\int y\,dy = \int (2\cos x + 1) \cdot dy$$

$$\frac{y^2}{2} = 2\sin x + x$$

$$y = \sqrt{4\sin x + 2x + c}$$

DIFFERENTIAL EQUATION WITH BOUNDARY CONDITION

Solve $\frac{dx}{dt} = t$: (a) $t = 0$, $x = 3$

$$\int_3^x dx = \int_0^t t \cdot dt$$

$$[x]_3^x = \left[t^2/2\right]_0^t$$

$$x - 3 = \frac{t^2}{2} - \frac{0^2}{2}$$

$$2x - 6 = t^2$$

$$t = \sqrt{2x - 6}$$

Solve the boundary line values:

(1) $\frac{dx}{dt} = \cos t$: (a) $t = 0$, $x = 5$

$$\int_5^x dx = \int_0^t \cos t \cdot dt$$

$$[x - 5] = \sin t$$

$$x = \sin t + 5$$

(2) $\frac{dv}{dt} = a$: (a) $t = 0$, $v = u$

$$\int^v dv = \int^t a\,dt$$

$v - u = at$

$$\boxed{v = u + at}$$

(3) $\frac{ds}{dt} = u + at$: (a) $t=0$, $s=0$

$$\int_0^s ds = \int_0^t (u + at)\,dt$$

$$s = \left[ut + \frac{at^2}{2}\right]_0^t$$

$$\boxed{s = ut + \frac{1}{2}at^2}$$

(4) $v \cdot \frac{dv}{ds} = a$: (a) $s=0$, $v=u$

$$\int_u^v v \cdot dv = \int_0^s a \cdot ds$$

$$\left[\frac{v^2}{2}\right]_u^v = \left[a \cdot s\right]_0^s$$

$$\frac{v^2}{2} - \frac{u^2}{2} = as$$

$$\boxed{v^2 - u^2 = 2as}$$

INSTANTANEOUS VELOCITY

The average velocity for an infinitly small decimal interval of time in the viscinity of given instant of time is called instantaneous velocity at that time. Thus if ds be a small displacement in a small time interval dt, then the instantaneous velocity at that instant is given by

$$\vec{V}_{in} = \frac{d\vec{s}}{dt}$$

Q.) The displacement of a particle is given by $s = 3t^2 + 2$. Find its velocity at $t = 3s$.

$$v = \frac{ds}{dt} = 6t$$
$$= 6 \times 3$$
$$= 18 \text{ m/s}$$

NOTE: By default it is assumed that at $t = 0$, $s = 0$

Q. The velocity of a particle is given by $v = 2\cos t$. Find S

$$S = \frac{dv}{dt} = \frac{d}{dt}(2\cos t)$$

$$S = 2\sin t$$

AVERAGE ACCELERATION

The ratio of change in velocity in a given interval of time to the duration of that interval is called average acceleration.

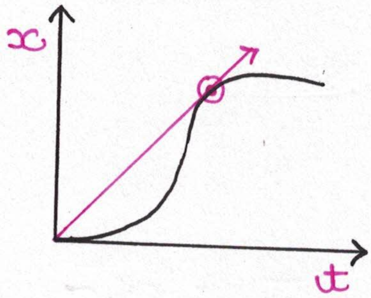
Mathematically,

If $\Delta\vec{v}$ be the change in velocity in a time interval Δt,

$$\vec{a}_{avg} = \frac{\Delta\vec{v}}{\Delta t}$$

NOTE: Note that for any trajectory, the avg. velocity is along the chord of the trajectory and the instantaneous velocity is along the tangent.

Q. At which point is the avg. velocity in the same direction as the instantaneous velocity.



AVERAGE SPEED (Vavg.)

The ratio of the length of the path covered in a time interval to duration of that time interval is called average speed, during that time interval.

Mathematically,

$$\vec{V}_{avg.} = \frac{\Delta l}{\Delta t}$$

NOTE: By the definitions it is clear that magnitude of avg. velocity is always less then or equal to avg. speed.

INSTANTANEOUS SPEED (v)

Let dl be the small distance covered by a particle in a small time interval dt then instantaneous speed is given by

$$v = \frac{dl}{dt}$$

NOTE: When the time interval becomes very small in the limit, the chord length approaches the arc length i.e. $dl = dS$. Thus the instantaneous speed is same as the magnitude of instantaneous velocity.

INSTANTANEOUS ACCELERATION

The average acceleration for an infinite decimal interval of time is called the instantaneous acceleration.

Mathematically,

$$\vec{a}_{in} = \lim_{\Delta t \to 0} \frac{\Delta \vec{v}}{\Delta t}$$

$$\vec{a}_{in} = \frac{d\vec{v}}{dt}$$

In case of motion along a straight line, we generally omit the vector sign and write,

$$a_{in} = \frac{dv}{dt}$$

DIFFERENTIAL EQUATIONS GOVERNING MOTION ALONG A STRAIGHT LINE

$$\boxed{v = \frac{ds}{dt}} \quad , \quad \boxed{a = \frac{dv}{dt}}$$

$$a = \frac{dv}{dt} \times \frac{ds}{dt} = \frac{dv}{ds} \times \frac{ds}{dt}$$

$$\boxed{a = v . \frac{dv}{ds}}$$

If a particle is moving along straight line, say x-axis, then its displacement at a general time is given by

$$S = x - x_0$$

$$\frac{ds}{dt} = \frac{dx}{dt} - 0$$

$$\boxed{v = \frac{dx}{dt}}$$

Using the basic differential equations prove the basic equations of motion along the straight line with uniform acceleration (assume @ $t=0$, $s=0$, $v=u$, $x=x_0$)

1.) $v = u + at$

2.) $x = ut + \frac{1}{2}at^2$

3.) $v^2 - u^2 = 2as$

4.) $x = x_0 + ut + \frac{1}{2}at^2$

$$v = \frac{dx}{dt}$$

$$\int_0^t u + at.\, dt = \int_{x_0}^{x} dx$$

$$\left[ut + \frac{1}{2}at^2\right]_0^t = x - x_0$$

$$x = x_0 + ut + \frac{1}{2}at^2$$

GENERAL MOTION WITH UNIFORM ACCELERATION

1.) $\vec{S} = \vec{u}t + \frac{1}{2}\vec{a}t^2$

2.) $\vec{v} = \vec{u} + \vec{a}t$

3.) $v^2 - u^2 = 2as$

$\vec{v}.\vec{v} - \vec{u}.\vec{u} = 2\vec{a}.\vec{s}$

4.) $\vec{r} = \vec{r_0} + \vec{u}t + \frac{1}{2}\vec{a}t^2$

DISPLACEMENT IN n^{th} SECOND

1.) $\vec{S_n} = \vec{u}n + \frac{1}{2}\vec{a}n^2$

2.) $\vec{S}_{n-1} = \vec{u}(n-1) + \frac{1}{2}\vec{a}(n-1)^2$

3.) $\vec{S}_{n^{th}} = \vec{u} + \frac{1}{2}\vec{a}(2n-1)$

LINEAR MOTION UNDER GRAVITY

1.) $\vec{S} = \vec{u}t - \frac{1}{2}gt^2$

2.) $\vec{v} = \vec{u} - gt^2$

3.) $\vec{v}^2 = \vec{u}^2 - 2gh$

Q.) A ball is projected vertically upwards with speed u. Find

(i) Maximum height

(ii) Time to reach max. height

(iii) Speed at half the max. height

(i) max. height $= \frac{u^2}{2g}$

(ii) time $= \frac{u}{g}$

(iii) Speed $v = \sqrt{u^2 - \frac{u^2}{2}}$ | $v = \sqrt{u^2 - gH}$

$= \frac{u}{\sqrt{2}}$

Q.) A ball is dropped from a balloon going up at a speed u when it is at a height H. How long will the ball take to reach the ground?

$$-H = ut - \frac{1}{2}gt^2$$

$$gt^2 - 2ut - 2H = 0$$

$$t = \frac{2u \pm \sqrt{4u^2 + 8gH}}{2g}$$

$$t = \frac{u \pm \sqrt{u^2 + 2gH}}{g}$$

Q.) A stone is thrown upwards with an unknown speed. What is its speed before reaching max. height.

$v = u - g(1)$ $\{u=0\}$

$v = -g$

$v = \cancel{-}g$

or

$v = u - gt$

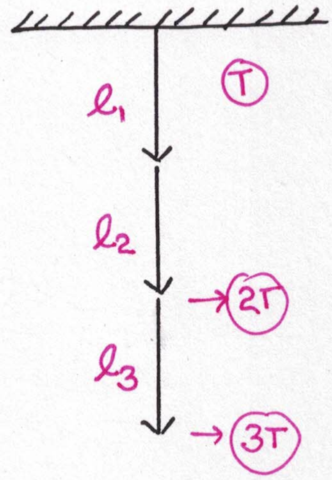
$v = u - g(u/g - 1)$

$v = \cancel{u} - \cancel{u} + g$

$v = +g$

Q.) Drops are falling at a regular interval from a roof as shown in the fig. Find the ratio $l_1 : l_2 : l_3$ when another drop was just going to fall.

$S_1 = -\frac{1}{2}gT^2$



$S_2 = -\frac{1}{2}g4T^2$

$S_3 = -\frac{1}{2}g9T^2$

$l_1 = -\frac{1}{2}gT^2$, $l_2 = -\frac{1}{2}g3T^2$

$l_3 = -\frac{1}{2}g5T^2$

$l_1 : l_2 : l_3 = 1 : 3 : 5$

Q.) A ball is dropped from an unknown height. It takes t_o seconds to cover the last H meters of its fall. Find the height from which it was dropped. ($t_o = 0,2$) & ($H = 6$)

$$v^2 = 2gh$$

$$v = \sqrt{2gh}$$

$$-H = \sqrt{2gh}\, t_o - \frac{1}{2} g t^2$$

$$-6 = \sqrt{20h} \times \frac{2}{10} - \frac{1}{2} \times 10 \times \frac{2}{10} \times \frac{2}{10}$$

$$-\frac{\cancel{60}^{30}}{\cancel{2}} = \sqrt{20h} - 1$$

$$-30+1 = 20h$$

$$h = \frac{29 \times 29}{20} = 42.05$$

$$H + h = 48.05m$$

Q.) A ferry boat after shutting its engine moves according to the velocity $v = \frac{v_0 t_1^2}{t^2}$. How much will be the total distance travelled by boat. (t_1 is the time at which engine is shut.)

$$\int_0^S ds = \int_4^\infty \frac{v_0 t_1^2}{t^2} . dt$$

$$S = v_0 t_1^2 \left[-\frac{1}{t}\right]_{t_1}^{\infty}$$

$$S = v_0 t_1^2 \times \left[-\frac{1}{\infty} + \frac{1}{t_1}\right]$$

$$S = v_0 t_1$$

Q.) The velocity of a particle is given by $v = ae^{ks}$ where a and k are constants & s is displacement. How does acceleration depend on displacement.

$$\vec{a} = \frac{dv}{ds} . v$$

$$a = ae^{ks} \times k \times ae^{ks}$$

$$\vec{a} = a^2 e^{2ks} k$$

Q.) In the previous problem, find its displacement as fun. of time

$$v = a e^{ks}$$

$$ds/dt = a e^{ks}$$

$$\boxed{\int_0^S \frac{ds}{e^{-ks}} = \int_0^t a\cdot dt}$$

$$\int_0^S e^{-ks} = a\int_0^t dt$$

$$\left[\frac{e^{-ks}}{-k}\right]_0^S = at$$

$$e^{-ks} - 1 = -kat$$

$$e^{-ks} = 1 - kat \Rightarrow \log_e(1-kat) = -ks$$

$$s = \frac{\ln(1-kat)}{-k}$$

Q. In the previous problem, what is the acceleration as a func. of time.

$$\vec{a} = a^2 k\, e^{2ks}$$

$$\vec{a} = a^2 k\, e^{2\cancel{k}\left(\frac{\ln(1-kat)}{-\cancel{k}}\right)}$$

$$\vec{a} = a^2 k\, e^{\ln(1-kat)^{-2}}$$

$$\vec{a} = \frac{a^2 k}{(1-kat)^2}$$

Or

$$s = \frac{\ln(1-kat)}{-k}$$

$$v = \frac{ds}{dt} = \frac{-1}{\cancel{k}} \times \frac{1}{(1-kat)} \times -\cancel{k}a$$

$$\vec{a} = \frac{dv}{dt} = a\left[\frac{(1-kat)\times 0 - (-ka)}{(1-kat)^2}\right]$$

$$\vec{a} = \frac{a^2 k}{(1-kat)^2}$$

Q. $v = b\sqrt{kx}$. Find acceleration as a function of displacement.

$$\vec{a} = \frac{dv}{dx}\cdot v$$

$$\vec{a} = \frac{b\sqrt{k}}{2}(kx)^{-1/2} \times b\sqrt{kx}$$

$$\vec{a} = \frac{b^2 k}{2}$$

Q: Velocity of a particle is given by $v = 7t^2 - 5$. Find displacement as a function of acceleration.

$$v = \frac{ds}{dt} = 7t^2 - 5$$

$$\int_0^s ds = \int_0^t (7t^2 - 5).dt$$

$$s = \frac{7t^3}{3} - 5t$$

$$\vec{a} = \frac{dv}{dt} = 14t$$

$$t = \frac{a}{14} \Rightarrow s = \frac{7}{3}\left(\frac{a^3}{14^3}\right) - 5\left(\frac{a}{14}\right)$$

Q: Acceleration of a particle varies as $a = -K\sqrt{v}$. At the initial moment, the velocity of particle is v_0. How much distance will it cover before stopping and how much time will it take to cover that distance.

$$a = v.\frac{dv}{ds}$$

$$\int_0^s ds = \int_{v_0}^{0} \frac{\sqrt{v}}{-K\sqrt{v}}.dv$$

$$S = -\frac{1}{K}\left[\frac{v^{3/2}}{3}\right]_{v_0}^{0}$$

$$S_f = -\frac{1}{K}\left[-\frac{2}{3}v_0^{3/2}\right]$$

$$S_f = \frac{2\sqrt{v_0^3}}{3K}$$

$$a = \frac{dv}{dt}$$

$$\int_0^t dt = \int_{v_0}^{a} \frac{1}{a}.dv$$

$$t = \int_{v_0}^{a} -\frac{1}{K\sqrt{v}}.dv$$

$$t = \left[-\frac{1}{K}\frac{\sqrt{v}}{2}\right]_{v_0}^{0}$$

$$t = \frac{2\sqrt{v_0}}{K}$$

AREA OF A TRIANGLE

Area of a triangle

$$= \frac{1}{2} \times p\sin\theta \times q$$

$$= \frac{1}{2}(\vec{p} \times \vec{q})$$